(7 pages) Reg. No. : ....

Code No. : 6369 **Sub. Code : ZMAM 21**

M.Sc. (CBCS) DEGREE EXAMINATION, NOVEMBER 2022.

Second Semester

Mathematics – Core

ALGEBRA – II

(For those who joined in July 2021 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. Let J be the ring of integers, J_n, the ring of integers modulo n. Define $\phi : J \to J_n$ by $\phi(a) =$ remainder of a on division by n. Its Kernel $I(\phi)$ is

(a) $\{0\}$

(b) J

(c) the set of all multiples on n

(d) $\{0,1,2, n-1\}$

2. Which one of the following is a maximal ideal of the ring of integers?

(a) (60) (b) (13)

(c) (2022) (d) (15)

3. A solution of the congruence $x^2 \equiv -1 (\text{mod } 13)$ is

(a) 3 (b) 0

(c) 5 (d) 6

4. A necessary and sufficient condition that the element a is the Euclidean ring be a unit is that

(a) $d(a) = 1$

(b) $d(a) = d(1)$

(c) $d(a) \mid d(1)$

(d) a is a prime element

5. Which one of the following is a primitive polynomial

(a) $2 + 4x^2 + 8x^5$

(b) $5 + 10x^2 + 15x^4$

(c) $20x^4 + 15x^3 + 10x^2 + 1$

(d) $2 + 2x + 2x^2$

6. Which one of the following is not true the polynomial $x^2 + 1$ is irreducible over?

(a) the complex field

(b) the real field

(c) the integers mod 3

(d) the field of rational numbers

7. In the ring of integers z, $\sqrt{(180)}$ is

(a) (4.9.5) (b) (2.3.5)

(c) (2.3.10) (d) (3.5.7)

8. In any ring R, an element $a \in rad\, R$ if and only if

(a) $1 - ra$ is invertible for each $r \in R$

(b) ra is invertible for each $r \in R$

(c) $1 - ra$ is invertible for some $r \in R$

(d) ra is invertible for some $r \in R$

9. b is a quasi-inverse of a if

(a) $ab = 1$ (b) $a + b - ab = 0$

(c) $a + b = 0$ (d) $a - b - ab = 0$

10. A ring R is isomorphic to a subdirect sum of integral domains if and only if R is

(a) semi simple

(b) simple

(c) without prime radical

(d) an integral domain

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) If F is a field, prove its only ideals are (0) and F itself.

Or

(b) If ϕ is a homomorphism of R into R', prove that $\phi(0) = 0$ and $\phi(-a) = -\phi(a)$ for every $a \in R$.

12. (a) Let R be a Euclidean ring and let A be an ideal of R. Prove that there exists an elements $a_0 \in A$ such that A consists exactly of all a_0x as x ranges over R.

Or

(b) Prove that $J[i]$ is a Euclidean ring.

13. (a) If $f(x), g(x)$ are two non zero elements of $F[x]$, prove that $\deg(f(x)g(x)) = \deg f(x) + \deg g(x)$.

Or

(b) State and prove Gauss' lemma.

14. (a) Let I be an ideal of the ring R. Prove that $I \subseteq rad\,R$ if and only if each element of the coset $1+I$ has an inverse in R.

Or

(b) For any ring R, prove that the quotient ring $R \mid Rad\,R$ is without prime radical.

15. (a) Define the J-radical $J(R)$ of a ring and a J-semi simple ring. Prove that the ring of even integers is J-semi simple.

Or

(b) Prove that a ring R is isomorphic to a sub direct sum of ring R_i, if and only if R contains a collection of ideals $\{I_i\}$ such that $R/I_i \simeq R_i$ and $\bigcap I_i = \{0\}$.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) If U is an ideal of the ring R, prove that $R \mid U$ is a ring and is a homomorphic image of R.

Or

(b) Let R be a commutative ring with unit element whose only ideals are (0) and R itself. Prove that R is a field.

17. (a) Prove that the ideal $A = (a_0)$ is a maximal ideal of the Euclidean ring R is and only if a_0 is a prime element of R.

Or

(b) Let p be a prime integer and suppose that for some integer c relatively prime to p we can find integers x and y such that $x^2 + y^2 = cp$. Prove that p can be written as the sum of squares of two integers.

18. (a) Define the sum $p(x)+q(x)$ and product $p(x)\cdot q(x)$ of two polynomials and state and prove the division algorithm for polynomials.

Or

(b) State and prove the Eisenstein criterion.

19. (a) If I is an ideal of the ring R, prove that

(i) $rad(R/I) \supseteq \dfrac{radR + I}{I}$ and

(ii) Whenever $I \subseteq radR$, $rad(R/I) = (rad\,R)/I$

Or

(b) (i) For any ring R, prove that $rad R[x] = Rad R[x]$.

(ii) Let e and e' be two idempotent elements of the ring R such that $e - e' \in Rad\, R$ prove that $e = e'$.

20. (a) If R is a ring such that $J(R) \neq R$, then prove that $J(R) = \bigcap \{M \mid M$ is a modular maximal ideal of $R\}$.

Or

(b) Prove that a ring R is isomorphic to a sub direct sum of fields it and only if for each non zero ideal I of R, then exists an ideal $J \neq R$ such that $I + J = R$.
